Math 1320: Systems of Linear Equations in Two Variables

What is a system of linear equations? We have seen linear equations in the form $y = mx + b$, whose graph is a straight line. The same graph could also be represented by $Ax + By = C$. A system of linear equations is when we have two or more linear equations working together, like the example below. For now we will only be working with systems that have two variables (x, y).

$$\begin{cases} x + 2y = 8 \\ 2x + 3y = -2 \end{cases}$$

Why is solving a system of linear equations important? A solution to a linear system is an ordered pair that makes all equations in the system true. Linear systems may have exactly one solution, no solution, or infinitely many solutions.

Linear systems are a tool for modeling real-life situations. Learning the skills to solve linear systems in two variables will help us to solve systems of equations in three variables and non-linear equations. There are many different methods for solving linear systems, we'll look at the substitution and addition methods.

How do we solve a system of linear equations?

1. Substitution Method
   - This method is most useful if one of the given equations has an isolated variable, or if an equation has a variable that can be easily solved for.
2. Addition Method
   - This method can be used for any linear system, just like the substitution method. This method is the most commonly used in solving linear systems.
3. Solutions
   - Note, when we use these methods it's possible to have none, one, or many solutions. We may check these graphically, as we will see later.

Substitution Method

Solving Linear Systems by Substitution	
Step 1	Solve either of the equations for one variable in terms of the other. (If one of the equations is already in this form, you can skip this step.)
Step 2	Substitute the expression found in step 1 into the *other* equation. This will result in an equation in one variable.
Step 3	Solve the equation containing one variable.
Step 4	Back-substitute the value found in step 3 into one of the original equations. Simplify and find the value of the remaining variable.
Step 5	Check the proposed solution in both of the system's given equations.

Example 1. Solve by the substitution method: $\begin{cases} x + 3y = 6 & \text{(Eq.1)} \\ 2x + 8y = -12 & \text{(Eq.2)} \end{cases}$

Step 1: Solve Eq. 1 for x.

$x + 3y = 6$	Eq. 1
$x = 6 - 3y$	Subtract $3y$ from both sides to solve for x

Step 2: Substitute $6 - 3y$ for x in Eq. 2.

$2x + 8y = -12$	Eq. 2
$2(6 - 3y) + 8y = -12$	Replace x with $6 - 3y$

Step 3: Solve for y.

$2(6 - 3y) + 8y = -12$	Equation from step 2
$12 - 6y + 8y = -12$	Apply distributive property
$12 + 2y = -12$	Combine like terms: $-6y + 8y = 2y$
$2y = -24$	Subtract 12 from both sides
$y = -12$	Divide both sides by 2

Step 4: Use $y = -12$ to find the value of x.

Using Equation 1	**Using Equation 2**
$x + 3y = 6$	$2x + 8y = -12$
$x + 3(-12) = 6$	$2x + 8(-12) = -12$
$x - 36 = 6$	$2x - 96 = -12$
$x = 42$	$2x = 84$
	$x = 42$

Notice that no matter what equation we choose to substitute -12 for y into, we get the same value for x. The solution is $(42, -12)$.

Step 5: Check the solution $(42, -12)$.

Does $(42, -12)$ make $x + 3y = 6$ true?
Does $(42, -12)$ make $2x + 8y = -12$ true?

Addition Method

Solving Linear Systems by Addition	
Step 1	If necessary, rewrite both equations in the form $Ax + By = C$.
Step 2	If necessary, multiply either equation or both equations by appropriate nonzero numbers so that the sum of the x-coefficients or the sum of the y-coefficients is 0.
Step 3	Add the equations in step 2. The sum is an equation in one variable.
Step 4	Solve the equation in one variable.
Step 5	Back-substitute the value obtained in step 4 into either of the given equations and solve for the other variable.
Step 6	Check the solution in both of the original equations.

Example 2. Solve by the addition method: $\begin{cases} 2x+5y=-2 & \text{(Eq.1)} \\ 6x+y=8 & \text{(Eq.2)} \end{cases}$

Step 1: Rewrite the equations in the form $Ax+By=C$

Both equations are already in this form.

Step 2: We can eliminate x or y. Let's eliminate x.

We can multiply each term of Eq. 1 by -3 and then add the resulting equations to eliminate x.

$$\begin{cases} 2x+5y=-2 \\ \quad 6x+y=8 \end{cases} \rightarrow \begin{cases} -6x-15y=6 \\ \quad 6x+y=8 \end{cases}$$

Step 3: Add the equations.

$$\begin{array}{r} \begin{cases} -6x-15y=6 \\ \quad 6x \quad + \quad y=8 \end{cases} \\ \hline -14y=14 \end{array}$$

Step 4: Solve equation from step 3.

$-14y=14$

$y=-1$ Divide both sides by -14

Step 5: Use $y=-1$ to find the value of x.

Using Equation 1	**Using Equation 2**
$2x+5y=-2$	$6x+y=8$
$2x+5(-1)=-2$	$6x+(-1)=8$
$2x-5=-2$	$6x=9$
$2x=3$	$x=\frac{9}{6}=\frac{3}{2}$
$x=\frac{3}{2}$	

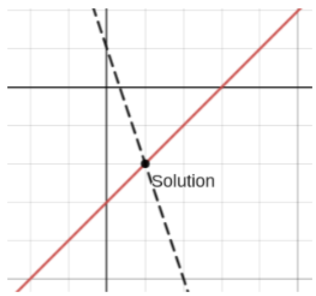
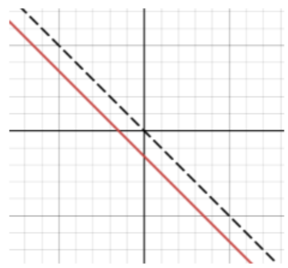
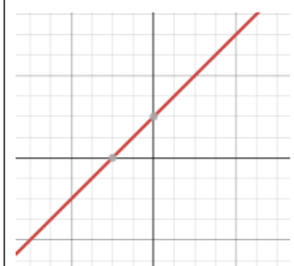
Notice that no matter what equation we choose to substitute -1 for y into, we get the same value for x. The solution is $(\frac{3}{2},-1)$.

Step 6: Check the solution $(\frac{3}{2},-1)$.

Does $(\frac{3}{2},-1)$ make $2x+5y=-2$ true?

Does $(\frac{3}{2},-1)$ make $6x+y=8$ true?

The Number of Solutions to a System of Two Linear Equations
In the examples above, there was exactly one solution that solved each linear system. There may be linear systems that give infinitely many solutions or no solutions. The table below gives examples for when these cases arise:

Number of Solutions	Exactly one ordered-pair solution	No solution	Infinitely many solutions
What This Means Algebraically	When solving such a system by substitution or addition, there is only one value that corresponds to each variable [e.g. $x = 2$ and $y = 5$]	When solving such a system by substitution or addition, you eliminate both variables and it results in a false statement [e.g. $0 = 4$]	When solving such a system by substitution or addition, you eliminate both variables and it results in a true statement [e.g. $0 = 0$]
What This Means Graphically	The two lines intersect at one point 	The two lines are parallel 	The two lines are identical 

Practice Problems Using the substitution or addition method, solve the following linear systems.

1. $\begin{cases} 3x + y = 7 \\ 9x + 3y = 21 \end{cases}$ [Infinitely many solutions $\to \{(x, y) \mid 3x + y = 7\}$ or $\{(x, y) \mid 9x + 3y = 21\}$]

2. $\begin{cases} \frac{1}{2}x = 1 + 5y \\ x = 2 + y \end{cases}$ [Exactly one solution $\to (2, 0)$]

3. $\begin{cases} 7x + 10y = 1 \\ -14x - 20y = 15 \end{cases}$ [No Solution]

4. $\begin{cases} x + 2y = 8 \\ 2x + 3y = -2 \end{cases}$ [Exactly one solution $\to (-28, 18)$]